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STABILITY THEOREMS FOR THE CONTINUOUS SPECTRUM OF A NEGATIVELY CURVED MANIFOLD

BY

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ABSTRACT. The spectrum of the Laplacian Δ for a simply connected complete negatively curved Riemannian manifold is studied. The Laplacian Δ_0 of a simply connected constant curvature space M_0 is known up to unitary equivalence. Decay conditions are given, on the metric g and curvature K of M, which imply that the continuous part of Δ is unitarily equivalent to Δ_0.

Introduction. Let M be a complete simply connected Riemannian manifold having negative sectional curvatures. Since M is complete, the Laplacian Δ of M is a selfadjoint unbounded operator on L^2M. When M is the symmetric space M_0 of constant negative curvature -1, then Δ_0 is known up to unitary equivalence, as summarized in §2. In particular, Δ_0 has purely absolutely continuous spectrum.

The present paper is concerned with stability of the continuous part of Δ_0 under perturbation of the metric g_0 on M_0. Theorem 4.12 gives decay conditions on the metric g and curvature K of M which guarantee that Δ has the same absolutely continuous part as Δ_0. Much weaker decay conditions on K alone guarantee that Δ has no singular continuous spectrum, as specified in Theorem 5.7. Combining these results, one obtains criteria which ensure that Δ has the same continuous part as Δ_0.

In our earlier paper [7], we established the above results for compactly supported perturbations of the metric on M_0. As well as obtaining stronger results, the current paper provides a better method, which may have other applications. One technique used essentially is to transplant the heat kernel and resolvent kernel from M_0 to M, as functions of the geodesic distance.

For background material on symmetric spaces and functional analysis, the reader may consult [7] and the references given there.

1. Heat kernels for complete Riemannian manifolds. The classical construction of a fundamental solution for the heat equation, as given in [**1**, pp. 204–215], uses repeatedly the hypothesis that one is working on a compact Riemannian manifold. However, as observed in [**4**, pp. 7–8] and [**5**, pp. 6–9], the usual method generalizes to any complete Riemannian manifold M having bounded geometry. Here we say that M has bounded geometry if its injectivity radius δ is bounded below and $\|\nabla^i R\|_\infty < D_i$, where $\nabla^i R$ is the ith covariant derivative of the curvature tensor R of M. In this section, we will show that the condition of bounded geometry can be

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weakened to assume only that the Ricci curvature and injectivity radius of M are bounded below.

Let Δ be the Laplacian for M. Since M is complete, the Laplacian Δ is an unbounded selfadjoint operator on L^2M. Therefore, the fundamental solution for the heat equation $\exp(-t\Delta)$; $L^2M \to L^2M$ is well defined by Hilbert space theory. We say that M has a good fundamental solution if $\exp(-t\Delta)$ is represented by a kernel $E(t, x, y)$ satisfying the properties:

P1. $(\partial/\partial t + \Delta_2)E(t, x, y) = 0$ where Δ_2 is the Laplacian acting in the second variable.

P2. $\lim_{t\to 0} E(t, x, y) = \delta(x, y)$, the Dirac delta measure.

P3. For $T > 0$ arbitrary and $0 \leqslant t \leqslant T$, one has when M is of dimension n:

$$|E(t, x, y)| \leqslant C_1 t^{-n/2} \exp\left(-C_2 r^2(x, y)/t\right)$$

where C_1, C_2 may depend only on T. Here r is the geodesic distance from x to y.

One has

THEOREM 1.1. *Let M be a complete negatively curved Riemannian manifold and Δ the Laplacian of M. Suppose that the Ricci curvature and injectivity radius of M are bounded below. Then the heat equation problem on M,*

$$(\partial/\partial t + \Delta)f(x, t) = 0, \qquad f(x, 0) = f_0(x),$$

has a good fundamental solution $E(t, x, y)$ satisfying properties P1–P3 *above.*

Furthermore $E(t, x, y)$ is unique and satisfies:

(Symmetry)

$$E(t, x, y) = E(t, y, x),$$

(Semigroup Property)

$$E(t + s, x, y) = \int_M E(t, x, z)E(s, z, y)\, dz.$$

PROOF. Let $\varepsilon < \delta$, where δ is the injectivity radius of M. Choose ϕ: $R \to R$ to be a smooth function satisfying $\phi(\alpha) = 1$ if $|\alpha| < \varepsilon/2$ and $\phi(\alpha) = 0$ if $|\alpha| > \varepsilon$. If r is the geodesic distance between points in M, then set $\eta(x, y) = \phi(r(x, y))$.

A first approximation for E is given by

$$E_1(t, x, y) = (4\pi t)^{-d/2} \exp(-r^2/4t)\eta(x, y).$$

For $\eta = 1$, this is just the fundamental solution for the heat equation on Euclidean space.

The proof requires some estimates concerning E_1 and related kernels:

LEMMA 1.2. *Denote $R_1 = (\partial/\partial t + \Delta_x)E_1$. Then one has the estimate*

$$|R_1(t, x, y)| \leqslant C_3 t^{-d/2-1/2} \exp\left(-C_4 r^2/t\right)$$

for $0 < t \leqslant T$, where C_3, C_4 may depend only on T.

PROOF. If $r(x, y) < \delta$, let $\theta(x, y)$ denote the volume element in spherical polar coordinates centered at x. Thus if $f \in C_0^\infty(B_\delta(x))$, where $B_\delta(x)$ is the ball of radius

δ about x, one has

$$\int_M f = \int f(r, \omega)\theta(r, \omega)\, dr\, d\omega$$

where (r, ω) are spherical polar coordinates centered at x.

When $f(r)$ is a function depending only on the geodesic distance from x to y, there is the well-known formula [**15**, p. 240]:

$$\Delta f = \frac{-d^2 f}{dr^2} - \frac{\theta'}{\theta}\frac{df}{dr} \tag{1.3}$$

with $\theta' = \partial\theta/\partial r$. Applying (1.3) and computing gives

$$R_1 = (4\pi t)^{-d/2}\exp(-r^2/4t)$$
$$\cdot\left[\eta\left(\frac{r}{2t}\right)\frac{\theta'}{\theta} + \left(\frac{r}{t}\right)\eta'(r) + \Delta\eta + \frac{1}{2}t^{-1}(1-d)\eta\right].$$

Since the Ricci curvature of M is bounded below and $r < \delta$, in the support of η, a standard comparison theorem [**3**, p. 253] gives $|\theta'/\theta + (1-d)/r| \leqslant B_1$. Note that $\theta'/\theta + (1-d)/r$ is exactly the logarithmic derivative in r of the Jacobian of the exponential map at x. Since η is the constant for $r < \varepsilon/2$, the same comparison theorem gives $|\Delta\eta| \leqslant B_2$. Thus

$$|R_1| \leqslant B_3 t^{-d/2-1} r\exp(-B_4 r^2/t).$$

So

$$|R_1| \leqslant B_3 t^{-d/2-1/2}\left[\frac{r^2}{t}\exp\left(\frac{-B_4 r^2}{2t}\right)\right]^{1/2}\exp\left(\frac{-B_4 r^2}{2t}\right).$$

The lemma now follows from the elementary inequality $\alpha e^{-k\alpha} \leqslant (ke)^{-1}$ applied to the quantity in brackets.

We may define

$$A * B(t, x, y) = \int_0^t ds\int_M A(s, x, z)B(t-s, z, y)\, dz$$

whenever the integrals converge absolutely. In this section only, one of the kernels A, B will be compactly supported in z for fixed x, y. Thus convergence of the integral over M is no problem. To show convergence of the t integral will require more careful estimates.

Denote $R_i = R_1 * R_1 * \cdots * R_1$ to be the i-fold convolution. Then we may write:

LEMMA 1.4. *For suitable constants* C_5, C_6 *one has*

$$|R_i(t, x, y)| \leqslant C_5 t^{(d/2+1/2)(i-2)}\exp(-C_6 r^2/t)$$

uniformly for $i \leqslant I$, $t \leqslant T$.

Moreover, for fixed x, $R_i(t, x, y)$ *has* y *support in* $B_{i\varepsilon}(x)$.

PROOF. Since $R_1(t, x, y) = 0$ if $r(x, y) \geqslant \varepsilon$, the definition of R_i shows that $R_i(t, x, y) = 0$ when $r(x, y) \geqslant i\varepsilon$.

Lemma 1.2 gives the desired estimate for R_1. Suppose, by induction, that we have shown

$$|R_{i-1}(t, x, y)| \leqslant B_5 t^{(d/2+1/2)(i-3)} \exp(-B_6 r^2/t).$$

Now, for $i \geqslant 2$,

$$R_i = R_1 * R_{i-1} = \int_0^t ds \int_M R_1(s, x, z) R_{i-1}(t - s, z, y)\, dz.$$

So

$$|R_i(t, x, y)| \leqslant \int_0^t ds\, C_3 B_5 s^{-d/2-1/2}(t - s)^{(d/2+1/2)(i-3)} \cdot \int_{\substack{r(x,z)<\varepsilon \\ r(y,z)<(i-1)\varepsilon}} \exp\left(\frac{-C_4 r^2(x, z)}{s}\right) \exp\left(\frac{-B_6 r^2(y, z)}{t - s}\right) dz.$$

Using the estimate,

$$r^2(x, y)/t \leqslant r^2(x, z)/s + r^2(y, z)/(t - s),$$

which follows from the triangle inequality, we may write

$$|R_i(t, x, y)| \leqslant B_7 \int_0^t s^{-1/2}(t - s)^{-1/2+(d/2+1/2)(i-2)}\, ds \exp(-B_8 r^2(x, y)/t).$$

Setting $s = t\lambda$, we find that

$$|R_i(t, x, y)| \leqslant B_7 \exp(-B_8 r^2(x, y)/t) t^{(d/2+1/2)(i-2)} \cdot \int_0^1 \lambda^{-1/2}(1 - \lambda)^{-1/2+(d/2+1/2)(i-2)}\, d\lambda.$$

The λ integral converges, so the lemma is established by induction.

For $i \geqslant 2$, the estimate of Lemma 1.4 shows that $R_i(t, x, y)$ extends continuously to $[0, \infty) \times M \times M$. Thus a convolution removes the singularity at $t = 0$ of R_1. One may now use the arguments of **[5]** to obtain the fundamental solution E on M.

Denote $S_{lj} = R_{2l+j}$ for $l \geqslant 0$ and $3 \leqslant j \leqslant 4$. Then one has

LEMMA 1.5. *For suitable constants C_7, C_8, C_9, independent of l, j, we have for* $0 \leqslant t \leqslant T$:

$$|S_{l,j}(t, x, y)| \leqslant \frac{C_7 C_8^l}{l!} t^{(d/2+1/2)(j-2)+l} \exp\left(\frac{-C_9 r^2(x, y)}{t}\right).$$

PROOF. Lemma 1.4 gives the result for $l = 0, j = 3, 4$. We proceed by induction on l:

$$S_{l,j} = S_{l-1,j} * R_2.$$

So from Lemma 1.4 and the induction hypothesis,

$$|S_{l,j}(t, x, y)| \leqslant \int_0^t \frac{C_7 C_8^{l-1}}{(l-1)!} s^{(d/2+1/2)(j-2)+l-1}\, ds \cdot C_5 \int_{\substack{d(z,y)<2\varepsilon \\ d(z,x)<(2l+j-2)\varepsilon}} \exp\left(\frac{-C_9 r^2(x, z)}{s}\right) \exp\left(\frac{-C_6 r^2(y, z)}{(t - s)}\right) ds.$$

Here we may suppose that $C_9 < C_6/2$. Using the elementary inequality $d^2(x,y)/t \leqslant d^2(x,z)/s + d^2(z,y)/(t-s)$:

$$|S_{l,j}(t,x,y)| \leqslant \int_0^t s^{(d/2+1/2)(j-2)+l-1}\,ds$$
$$\cdot \int C_7 \frac{C_8^{l-1}}{(l-1)!} C_5 \exp\left(\frac{-C_6 r^2(y,z)}{2T}\right) dz \exp\left(\frac{-C_9 r^2(x,y)}{t}\right).$$

Since the Ricci curvature is bounded below, the volume element grows at most exponentially, $\theta(y,z) \leqslant \exp(B_8 r(y,z))$ [3, p. 253]. Thus, the z integral is bounded.

So

$$|S_{l,j}(t,x,y)| \leqslant t^{(d/2+1/2)(j-2)+l} C_7 \frac{C_8^{l-1}}{l!} C_5 B_9 \exp\left(\frac{-C_9 r^2(x,y)}{t}\right).$$

This yields the estimate required by the lemma:

$$|S_{l,j}(t,x,y)| \leqslant \frac{C_7 C_8^l}{l!} t^{(d/2+1/2)(j-2)+l} \exp\left(\frac{-C_9 r^2(x,y)}{t}\right).$$

Now denote $Q = \sum_{i=1}^{\infty}(-1)^i R_i$. By Lemma 1.5, the series converges absolutely and one has

$$|Q(t,x,y)| \leqslant C_{10} t^{-d/2-1/2} \exp\left(\frac{C_9 r^2(x,y)}{t}\right)$$

when $0 \leqslant t \leqslant T$.

As in [1] and [5], a fundamental solution is obtained by setting $E = E_1 - E_1 * Q$. The uniqueness, semigroup, and symmetry properties of E follow, as in [5, p. 9], from Duhamel's principle.

This completes the proof of Theorem 1.1.

A crude estimate on the behavior of the heat kernel for large t is given by

Corollary 1.6. *Let M be as in Theorem 1.1. Then the heat kernel $E(t,x,y)$ satisfies the estimate*

$$|E(t,x,y)| \leqslant A_1 e^{A_2 t} t^{-n/2} \exp(-A_3 r^2(x,y)/t)$$

for some $A_1, A_2, A_3 > 0$.

Proof. Theorem 1.1 and property P3 give the required estimate for $t \leqslant T$ and any $T > 0$. It suffices to show that, for large t, one has

$$|E(t,x,y)| \leqslant A_1 e^{A_4 t} \exp(-A_3 r^2(x,y)/t). \tag{1.7}$$

By property P3 we may write

$$|E(1,x,y)| \leqslant C_1 \exp(-C_2 r^2(x,y)).$$

Assume by induction that (1.7) holds for $t \leqslant T$ and some $A_3 \leqslant C_2/2$, $T \geqslant 2$. Let $T < t \leqslant T+1$.

The semigroup property reads

$$E(t+1,x,y) = \int E(t,x,z) E(1,z,y)\,dz.$$

So

$$|E(t+1,x,y)| \leq C_1 A_1 e^{A_4 t} \int \exp\left(\frac{-A_3 r^2(x,z)}{t}\right) \exp(-C_2 r^2(z,y))\, dz.$$

Since $r^2(x,y)/(t+1) \leq r^2(x,z)/t + r^2(z,y)/2$, we have

$$|E(t+1,x,y)| \leq C_1 A_1 e^{A_4 t} \int \exp\left(\frac{-C_2 r^2(z,y)}{2}\right) dz \, \exp\left(\frac{-A_3 r^2(x,y)}{t+1}\right).$$

Thus

$$|E(t+1,x,y)| \leq A_1 e^{A_4(t+1)} \exp(-A_3 r^2(x,y)/(t+1)).$$

This completes the induction and proof of the corollary.

2. The constant curvature case. Let M be a complete simply connected Riemannian manifold having constant curvature -1. The Laplacian Δ of M is identified, up to unitary equivalence, by the theory of special functions on M. These constant curvature spaces will be used as models in the present paper.

If M is of dimension n, then the Laplacian Δ has purely absolutely continuous spectrum supported on the half line $[(n-1)^2/4, \infty)$. Let $L^2(R^+, dx, \mathfrak{M})$ denote the space of square Lebesgue integral $\mathfrak{M}$-valued functions on the positive real line. Here $\mathfrak{M}$ is a Hilbert space of countable infinite dimension. It is well known **[11**, pp. 109, 131] that Δ is unitarily equivalent to the multiplication operator $f(x) \to [(n-1)^2/4 + x^2]f(x)$, for $f \in L^2(R^+, dx, \mathfrak{M})$.

The spherical transform of Harish-Chandra **[11]** may be employed to obtain formulas representing the heat kernel and resolvent kernel of Δ. However, for our purposes, a more elementary approach will suffice.

According to Theorem 1.1, M has a good heat kernel $E(t, x, y)$. In fact, the Hadamard Cartan Theorem [**3**, p. 184] implies that M has infinite injectivity radius. Furthermore, M is a symmetric space and therefore admits a transitive group of isometries G. Uniqueness of the heat kernel gives $E(t, gx, gy) = E(t, x, y)$. Moreover, since M has rank one, the isotropy group at each $x \in M$ is transitive on the unit sphere in $T_x M$, so $E(t, x, y) = E(t, r(x, y))$. Here $r(x, y)$ is the geodesic distance from x to y. For background on symmetric spaces, the reader may consult [**10**].

The resolvent equation $(\Delta - z)f = 0$ has a fundamental solution $R(z, x, y)$, analogous to the heat kernel. In fact, if z lies in some left half-plane, Re $z < -A_2$, then by Corollary 1.6, we may write

$$R(z,x,y) = \int_0^\infty e^{tz} K(t,x,y)\, dt. \tag{2.1}$$

When $x \neq y$, (2.1) expresses the resolvent kernel as the Laplace transform of the heat kernel. In particular, the kernel $R(z, x, y)$ exists for Re $z < -A_2$ and $R(z, x, y) = R(z, r(x, y))$.

Suppose Re $z < -A_2$. Then since R is a function of r alone we have [**15**, p. 240]

$$\Delta R = \frac{-d^2 R}{dr^2} - \frac{\theta'}{\theta} \frac{dR}{dr}.$$

In the constant curvature case [**3**, p. 253], $\theta = (\sinh r)^{n-1}$, so

$$(\Delta - z)R = \frac{-d^2R}{dr^2} - (n-1)\coth r\frac{dR}{dr} - zR = 0 \tag{2.2}$$

by definition of the resolvent $(\Delta - z)^{-1}$. Setting $x = \cosh r$, (2.2) becomes

$$(x^2-1)\frac{d^2R}{dx^2} + nx\frac{dR}{dx} + zR = 0. \tag{2.3}$$

Denote p to be the solution of $z = (n-1)^2/4 + p^2$ with p having positive imaginary part for $\operatorname{Re} z < -A_2$. Set $m = n/2 - 1$. Then the general solution of the ordinary differential equation (2.3) is of the form

$$R(z,x) = (x^2-1)^{-m/2}\left[a_1(z)P^m_{-1/2+\sqrt{-1}\,p}(x) + a_2(z)Q^m_{-1/2+\sqrt{-1}\,p}(x)\right] \tag{2.4}$$

where P, Q are the usual Legendre functions [**17**, I, pp. 65–67], for $x > 1$.

Since R represents the resolvent, for $\operatorname{Re} z < -A_2$, the coefficients a_1, a_2 are determined. In fact $R(z,x)$ must have the following properties: (i) $R(z,x)$ induces a bounded map $L^2M \to L^2M$, (ii) $R(z,x)$ has the same local singularity at $r = 0$ as the Euclidean Green's function. Using (i), (ii) and the standard asymptotic formulas for Legendre functions [**17**, II, pp. 14, 15, 75, 221, 222], one obtains explicit formulas representing a_1, a_2. The actual expressions are rather cumbersome. Our main point is that (2.4) provides a continuation of the kernel $R(z,x)$ from $\operatorname{Re} z < -A_2$ to the z-plane, with a branch cut along the interval $[(n-1)^2/4, \infty)$. The branch cut comes from extracting the square root $p = [z - (n-1)^2/4]^{1/2}$.

The special function theory also shows that the analytically continued kernel $R(z,r)$ induces a bounded map $L^2M \to L^2M$ for $z \in [(n-1)^2/4, \infty)$. Thus, by the uniqueness of analytic continuation, $R(z,r)$ must represent the resolvent $(\Delta - z)^{-1}$ for $z \in C - \operatorname{Spec}\Delta$.

3. Transplanted heat kernels. Let M be a complete simply connected n-dimensional Riemannian manifold having negative sectional curvatures. By a theorem of Hadamard and Cartan [**3**, p. 183], the exponential map $\exp: T_pM \to M$ is a diffeomorphism for each $p \in M$. Consequently, there is a system of spherical polar coordinates (r, ω) about p, with volume element $\theta(r,\omega)$. If M_0 is the simply connected complete space having constant curvature -1, then $\theta_0 = (\sinh r)^{n-1}$, independent of p, ω.

Suppose that the metric on M is obtained by perturbing the metric of M_0. We would like to give decay conditions on the metric g and curvature K of M which guarantee that the Laplacian Δ of M has the same absolutely continuous part as the Laplacian Δ_0 of M_0. This section provides a technical device for attacking the problem of stability for the absolutely continuous spectrum. The main idea is to transplant the heat kernel E_0 from M_0 to M by regarding $E_0(t,r)$ as a function of the geodesic distance r on M.

Let $E_0(t)$ be the heat kernel of M_0 for fixed $t > 0$. Recall from §2, that $E_0(t)$ depends only upon the geodesic r_0 between points in M_0. Consequently, we may define $F(t,x,y) = E_0(t, r(x,y))$, where $r(x,y)$ is the geodesic distance in M.

Property P3 of §1 gives the estimate $|F(t, x, y)| \leq C_1 \exp(-C_2 r^2(x, y))$ for fixed $t > 0$.

By using the exponential maps at p, we may identify the differentiable manifolds underlying M, M_0. Suppose that, modulo this identification, the metric g satisfies the decay conditions

$$(1 + \beta)^{-2} g_0(V, V) \leq g(V, V) \leq (1 + \beta)^2 g_0(V, V) \tag{3.1}$$

for $V \in T_x M$. Here $\beta(x) = D_1 \exp(-D_2 r(x, p))$, with $D_2 > 0$. For convenience, denote $\gamma(x) = r(x, p)$. Using (3.1), we see that $|\theta(p, x)/\theta_0(\gamma(x))|$ is bounded above and below by positive constants. This allows one to identify $L^2 M$ and $L^2 M_0$, via geodesic spherical coordinates about p. Moreover, the kernel F induces bounded selfadjoint operators $F_0(t)$: $L^2 M_0 \to L^2 M_0$ and $F(t)$: $L^2 M \to L^2 M$, which are unitarily equivalent.

We first observe

LEMMA 3.2. *Suppose that in* (3.1), $\beta(\gamma) = D_3 \exp(-D_4 \gamma)$ *with* $D_4 > n - 1$. *Then* $F_0(t) - E_0(t)$ *is Hilbert-Schmidt.*

PROOF. Let r_0 denote the geodesic distance in M_0. Then $r_0(p, x) = r(p, x) = \gamma(x)$. The difference $P(t) = F_0(t) - E_0(t)$ has kernel $P(t, x, y) = E_0(t, r(x, y)) - E_0(t, r_0(x, y))$.

Choose $\varepsilon < 1$ so that $\varepsilon D_4 > n - 1$. Then if $r(x, y) \leq (1 - \varepsilon)\gamma(x)$, the triangle inequality yields $\gamma(y) > \varepsilon\gamma(x)$. Consequently, $[1 + \beta(\varepsilon\gamma)]^{-1} r_0 \leq r \leq [1 + \beta(\varepsilon\gamma)] r_0$, where $\gamma = \gamma(x)$.

Clearly

$$|P(t, x, y)| \leq \int_{r_0}^{r} \left| \frac{\partial}{\partial r} E_0 \right| dr.$$

However, for fixed t, it is well known [5, pp. 6–9] that $|\partial E_0/\partial r| \leq C_3 \exp(-C_4 r^2)$. So

$$|P(t, x, y)| \leq C_3 \exp\left(-C_5 r_0^2(x, y)\right)(r - r_0).$$

Thus

$$|P(t, x, y)| \leq C_6 \beta(\varepsilon\gamma(x)) \exp\left(-C_5 r_0^2/2\right).$$

By applying the triangle inequality, we deduce that

$$|P(t, x, y)| \leq C_7 \beta(\varepsilon\gamma(x)/2) \beta(\varepsilon\gamma(y)/2) \exp\left(-C_5 r_0^2(x, y)/4\right) \tag{3.3}$$

if $r(x, y) \leq (1 - \varepsilon)\gamma(x)$. By symmetry, one has (3.3) when $r(x, y) \leq (1 - \varepsilon)\gamma(y)$.

Now suppose that $r(x, y) \geq \max((1 - \varepsilon)\gamma(x), (1 - \varepsilon)\gamma(y))$. Then using $|P(t, x, y)| \leq |F(t, x, y)| + |E_0(t, x, y)|$ we see that

$$|P(t, x, y)| \leq C_8 \beta(\varepsilon\gamma(x)) \beta(\varepsilon\gamma(y)). \tag{3.4}$$

Using (3.3), (3.4) and the condition $\varepsilon D_4 > n - 1$, we find

$$\int_{M_0 \times M_0} [P(t, x, y)]^2 \, dx \, dy < \infty.$$

So $P(t)$ is Hilbert-Schmidt.

Now let

$$G_0(2t, x, y) = \int_{M_0} F(t, x, z) F(t, z, y)\, dz,$$

so that $G_0(2t) = F_0(t) \circ F_0(t)$, the composition. Of course, $G_0(2t)\colon L^2M_0 \to L^2M_0$ is a bounded selfadjoint operator. Moreover, we have

PROPOSITION 3.5. *Suppose that in* (3.1), $\beta = D_3 \exp(-D_4\gamma(x))$ *with* $D_4 > n - 1$. *Then* $E_0(2t) - G_0(2t)$ *is trace class.*

PROOF. If $\varepsilon < 1$, so that $\varepsilon D_4 > n - 1$, let $\mathfrak{M}$ be the operator of multiplication by $\exp(\varepsilon D_4 \gamma(x)/2)$. Employing the factorization trick of [**13**, p. 1190] we write

$$\begin{aligned} E_0(2t) - G_0(2t) &= \left[E_0(t)\mathfrak{M}^{-1}\right]\left[\mathfrak{M}(E_0(t) - F_0(t))\right] \\ &\quad + \left[(E_0(t) - F_0(t))\mathfrak{M}\right]\left[\mathfrak{M}^{-1}F_0(t)\right], \end{aligned}$$

using the semigroup property of $E_0(t)$. As in Lemma 3.2, the inequalities (3.3) and (3.4) imply that each operator in brackets is Hilbert-Schmidt. So $E_0 - G_0$ is a trace class.

The main result of this section is

THEOREM 3.6. *Let M be a complete simply connected negatively curved manifold whose metric is obtained by perturbing the metric g_0 of the constant curvature space M_0. Suppose that the metric g of M satisfies the decay condition* (3.1) *with* $D_4 > n - 1$.

Denote by $F(t)\colon L^2M \to L^2M$ the selfadjoint operator obtained by transplanting $E_0(t)$ via $F(t, x, y) = E_0(t, r(x, y))$, where r is the geodesic distance on M. Then, for any $t > 0$, the absolutely continuous part of $F(t)\colon L^2M \to L^2M$ is unitarily equivalent to $E_0(t)\colon L^2M_0 \to L^2M_0$.

PROOF. We have observed that $F(t)$ is unitarily equivalent to $F_0(t)\colon L^2M_0 \to L^2M_0$. By Proposition 3.5, $E_0(2t) = E_0(t) \circ E_0(t)$ has $G_0(2t) = F_0(t) \circ F_0(t)$ as a trace class perturbation. Thus $E_0(2t)$ and $G_0(2t)$ have the same absolutely continuous part by a theorem of Birman and Kato [**2**, p. 98]. Theorem 3.6 now follows by extracting the positive square roots $E_0(t)$, $F_0(t)$ of $E_0(2t)$, $F_0(2t)$.

4. The absolutely continuous spectrum. Let us continue in the framework of §3. We have shown that the operator $F(t)\colon L^2M \to L^2M$ with kernel $F(t, x, y) = E_0(t, r(x, y))$ has absolutely continuous part which is unitarily equivalent to $\exp(-t\Delta_0)\colon L^2M_0 \to L^2M_0$. In the present section, the kernel F will be employed as a parametrix to construct the fundamental solution $E(t, x, y)$ of the heat equation on M. Curvature decay conditions will be given which guarantee that $E(t)$ and $F(t)$ have the same absolutely continuous part.

In preparation, some technical lemmas are required:

LEMMA 4.1. *Let* $\lambda_1 \geqslant \lambda_2 > 0$. *Then*
(i) $\sup_{s>0}(\lambda_1 \coth \lambda_1 s - \lambda_2 \coth \lambda_2 s) = \lambda_1 - \lambda_2$;
(ii) $\sup_{s>0}(\lambda_1 \coth \lambda_1 s - 1/s) = \lambda_1$.

PROOF. Calculus.

Let $p \in M$ and suppose that $\gamma(x) = r(x, p)$ is the geodesic distance of x from p. Denote by $K(x, \pi)$ the sectional curvature of the two-plane π at x. Then one has

LEMMA 4.2. *Assume that for all (x, π), we have $|K(x, \pi) + 1| \leq C_1 \exp(-C_2\gamma(x))$, for $C_2 > 0$. Then, for $r(x, y) < (1 - \varepsilon)\gamma(x)$, $0 < \varepsilon < 1$, we may write*

$$|(\theta'/\theta)(x, y) - (\theta_0'/\theta_0)(r(x, y))| \leq D_3 \exp(-\varepsilon C_2\gamma(x))$$

where θ' is the partial derivative with respect to $r(x, y)$.

PROOF. By the triangle inequality, $\gamma(y) > \varepsilon\gamma(x)$. For simplicity, let us abbreviate $\gamma = \gamma(x)$ and $h(\gamma) = C_1 \exp(-C_2\gamma)$.

If $h(\varepsilon\gamma) < \frac{1}{2}$, then by a standard comparison theorem [3, p. 284]:

$$\begin{aligned} &|(\theta'/\theta)(x, y) - (\theta_0'/\theta_0)(r(x, y))| \\ &\quad \leq (n-1) \sup_{s \leq (1-\varepsilon)\gamma} \Big|\coth\big(\sqrt{1 + h(\varepsilon\gamma)}\, s\big)\sqrt{1 + h(\varepsilon\gamma)} \\ &\qquad\qquad - \coth\big(\sqrt{1 - h(\varepsilon\gamma)}\, s\big)\sqrt{1 - h(\varepsilon\gamma)}\Big|. \end{aligned}$$

So, by Lemma 4.1,

$$\begin{aligned} |(\theta'/\theta)(x, y) - (\theta_0'/\theta_0)(r(x, y))| &\leq (n-1)\big|\sqrt{1 + h(\varepsilon\gamma)} - \sqrt{1 - h(\varepsilon\gamma)}\big| \\ &\leq B_1 h(\varepsilon\gamma). \end{aligned}$$

On the other hand, if $h(\varepsilon\gamma) \geq \frac{1}{2}$, then by [3, p. 284] and Lemma 4.1,

$$\begin{aligned} &|(\theta'/\theta)(x, y) - (\theta_0'/\theta_0)(r(x, y))| \\ &\qquad \leq \sup_{s \leq (1-\varepsilon)\gamma} \big|\sqrt{1 + h(\varepsilon\gamma)}\, \coth\big(\sqrt{1 + h(\varepsilon\gamma)}\, s\big) - 1/s\big| \\ &\qquad \leq (n-1)\sqrt{1 + h(\varepsilon\gamma)} \leq B_2 h(\varepsilon\gamma) \end{aligned}$$

for $h(\varepsilon\gamma) \geq \frac{1}{2}$. This proves Lemma 4.2.

We may now state

THEOREM 4.3. *Let M be a complete simply connected n-dimensional Riemannian manifold having negative sectional curvatures. Fix $p \in M$, and suppose that for all (x, π), one has $|K(x, \pi) + 1| \leq C_1 \exp(-C_2\gamma(x))$. Here $\gamma(x)$ is the geodesic distance of x from p.*

If $C_2 > n - 1$, then the operators $\exp(-t\Delta)$ and $F(t)$: $L^2M \to L^2M$ have unitarily equivalent absolutely continuous part.

PROOF. We imitate the constructions of §1, using F as a parametrix to obtain a representation of the heat kernel E of M. Several lemmas are required. As observed in [6, p. 840], the curvature decay condition guarantees that $|\theta(p, x)/\theta_0(\gamma(x))|$ is bounded above and below by positive constants. Choose $0 < \varepsilon < 1$ so that $\varepsilon C_2 > n - 1$. Then one has

LEMMA 4.4. *Denote $R_1 = (\partial/\partial t + \Delta_x)F(t, x, y)$. Then for $0 \leq t \leq T$:*

$$|R_1(t, x, y)| \leq B_3 \exp\big(-C_2[\gamma(x) + \gamma(y)]\varepsilon/2\big)\exp\big(-B_4 r^2(x, y)/2t\big)t^{-n/2-1/2}$$

where B_3, B_4 depend only upon T.

PROOF. One has

$$\left(\frac{\partial}{\partial t}+\Delta\right)F(t,x,y)=\left[\frac{-\theta'}{\theta}(x,y)+\frac{\theta_0'}{\theta_0}(r(x,y))\right]\frac{\partial}{\partial r}E_0(t,r).$$

Now it is well known [5, pp. 6–9] that for $0<t\leqslant T$:

$$|\partial E_0(t,r)/\partial r|\leqslant D_4t^{-n/2-1/2}\exp\left(-B_4r^2(x,y)/t\right).$$

If $r(x,y)\leqslant(1-\varepsilon)\gamma(x)$, then by Lemma 4.2:

$$|R_1(t,x,y)|\leqslant D_3\exp(-C_2\varepsilon\gamma(x))D_4t^{-n/2-1/2}\exp\left(-B_4r^2(x,y)/t\right).$$

By the triangle inequality:

$$|R_1(t,x,y)|\leqslant\exp\left(-C_2[\gamma(x)+\gamma(y)]\varepsilon/2\right)\exp\left(-B_4r^2(x,y)/2t\right)D_5t^{-n/2-1/2}.$$

Similarly, Lemma 4.4 follows if $r(x,y)<(1-\varepsilon)\gamma(y)$.

Now suppose $r(x,y)\geqslant(1-\varepsilon)\gamma(x)$ and $r(x,y)\geqslant(1-\varepsilon)\gamma(y)$. Since the curvature of M is bounded below [3, p. 284],

$$|(\theta'/\theta)(x,y)-(\theta_0'/\theta_0)|\leqslant B_5.$$

So

$$|R_1(t,x,y)|\leqslant D_4B_5t^{-n/2-1/2}\exp\left(-B_4r^2(x,y)/2t\right)$$
$$\cdot\exp\left(-B_4[\gamma^2(x)+\gamma^2(y)](1-\varepsilon)^2/4t\right),$$

which establishes Lemma 4.4, when $r(x,y)\geqslant\max(\gamma(x),\gamma(y))(1-\varepsilon)$.

As in the proof of Theorem 1.1, we denote $R_i=R_1*R_1*\cdots*R_1$ to be the i-fold convolution.

We have

LEMMA 4.5. *For suitable constants* B_6, B_7, *one has*

$$|R_i(t,x,y)|\leqslant B_6t^{(d/2+1/2)(i-2)}\exp\left(-B_7r^2(x,y)/t\right)\exp\left(-C_2\varepsilon[\gamma(x)+\gamma(y)]/2\right)$$

uniformly if $i\leqslant I$, $t\leqslant T$.

PROOF. The proof is analogous to that of Lemma 1.4.

Define $S_{l,j}=R_{2l+j}$ for $l\geqslant 0$; derive

LEMMA 4.6. *For suitable constants* B_8, B_9, B_{10}, *independent of* l,j, *one has*

$$|S_{l,j}(t,x,y)|\leqslant\left(B_8B_9^l/l!\right)t^{(d/2+1/2)(j-2)+l}$$
$$\cdot\exp\left(-C_2\varepsilon[\gamma(x)+\gamma(y)]/2\right)\exp\left(-B_{10}r^2(x,y)/t\right).$$

PROOF. The proof is similar to the proof of Lemma 1.5.

If $Q=\sum_{i=0}^{\infty}(-1)^iR_i$, then the series converges absolutely and

$$|Q(t,x,y)|\leqslant B_{11}t^{-d/2-1/2}\exp\left(-B_{12}r^2(x,y)/t\right)\exp\left(-C_2\varepsilon[\gamma(x)+\gamma(y)]/2\right)$$

when $0\leqslant t\leqslant T$. Moreover, one has the estimate

$$|F*Q(t,x,y)|\leqslant A_1t^{1/2}\exp\left(-A_2r^2(x,y)/t\right)$$
$$\cdot\exp\left(-C_2\varepsilon[\gamma(x)+\gamma(y)]/2\right).\tag{4.7}$$

As in the proof of Theorem 1.1, we find that

$$E=F-F*Q.\tag{4.8}$$

It is now not difficult to show that $E - F$ defines a Hilbert-Schmidt operator. We have in fact,

LEMMA 4.9. *For any $t > 0$, the kernel*

$$P(t, x, y) = \exp(C_2\varepsilon\gamma(x)/2)[E(t, x, y) - F(t, x, y)]$$

defines a Hilbert-Schmidt operator.

PROOF. Using (4.7) and (4.8) we find that

$$\int_{M\times M}[P(t, x, y)]^2\, dx\, dy$$
$$\leq A_1^2 t\int \exp(-2A_2r^2(x, y)/t)\exp(-\varepsilon C_2\gamma(y))\, dx\, dy < \infty$$

since $\varepsilon C_2 > n - 1$.

Let $G = F^2$ be the composition

$$G(2t, x, y) = \int_M F(t, x, z)F(t, z, y)\, dz.$$

Then one has the estimate

$$|G(t, x, y)| \leq A_3 t^{-n/2}\exp(-A_4r^2(x, y)/t)$$

for $0 < t \leq T$. In general, $G(t) \neq F(t)$, since the measure on M is different from that of M_0, so the semigroup property of $\exp(-t\Delta_0)$ is lost. However, $G(t)$ is still unitarily equivalent to $\exp(-t\Delta_0)$.

We may state

LEMMA 4.10. *For any $t > 0$, the kernel $E(2t, x, y) - G(2t, x, y)$ defines a trace class operator.*

PROOF. As in [**13**, p. 1190] we exploit the semigroup property of E:

$$E(2t) - G(2t) = [E(t)\mathfrak{M}^{-1}][\mathfrak{M}(E(t) - F(t))]$$
$$+ [(E(t) - F(t))\mathfrak{M}][\mathfrak{M}^{-1}F(t)]$$

where $\mathfrak{M}$ is the multiplication operator $f(x) \to \exp(C_2\varepsilon\gamma(x)/2)f(x)$. A computation, similar to the proof of Lemma 4.9, shows that each operator in brackets is Hilbert-Schmidt.

Lemma 4.10 and the theorem of Birman and Kato [**2**, p. 98] imply that $E(2t)$ and $G(2t)$ have unitarily equivalent absolutely continuous part. Theorem 4.3 now follows by extracting the unique positive square roots $E(t)$ and $F(t)$ of $E(2t)$ and $G(2t)$.

REMARK 4.11. It would be logically flawless to omit §1 and to construct the heat kernel E on M directly from F as parametrix. However, such a presentation might be slightly misleading. The more general Theorem 1.1 also seems to have independent interest.

We may now collect the results of §§3 and 4 to state our main result on the absolutely continuous spectrum of Δ:

THEOREM 4.12. *Let M be a complete simply connected Riemannian manifold having negative sectional curvatures. Suppose that the metric of M is obtained by perturbation from the standard metric g_0 on the simply connected space of constant curvature* -1.

If g, K denote the metric and curvature of M, then we impose the decay conditions:

(i) $(1+\beta)^{-2}g_0(V,V) \leqslant g(V,V) \leqslant (1+\beta)^2 g_0(V,V)$ *for* $V \in T_xM$, *and*

(ii) $|K(x,\pi)+1| \leqslant h$ *for π a two plane at* $x \in M$.

Here $h(x) = C_1 \exp(-C_2\gamma(x))$ and $\beta(x) = C_3 \exp(-C_2\gamma(x))$. Moreover, $\gamma(x) = r(x,p)$ is the geodesic distance of x from a fixed $p \in M$. We assume that $C_2 > n-1$, where n is the dimension of M.

Under these conditions the absolutely continuous part of the Laplacian Δ: $L^2M \to L^2M$ is unitarily equivalent to Δ_0: $L^2M_0 \to L^2M_0$.

PROOF. By Theorem 3.6 and condition (i), the operators $F(t)$: $L^2M \to L^2M$ and $\exp(-t\Delta_0)$: $L^2M_0 \to L^2M_0$ have the same absolutely continuous part. However, $F(t)$ has the same absolutely continuous part as $\exp(-t\Delta)$ by condition (ii) and Theorem 4.3. Since Δ_0 is purely absolutely continuous, Theorem 4.12 follows.

REMARK 4.13. Theorem 4.12 is a considerable improvement over the corresponding result in the author's earlier paper [7, p. 3]. It was shown there that $\exp(-t\Delta) - \exp(-t\Delta_0)$ is trace class if the metric on M is obtained by a compactly supported perturbation of the metric on M_0. The method used there requires control over the higher order derivatives of the metric, while conditions (i) and (ii) only restrict the metric g and curvature K.

5. Singular continuous spectrum. Let M be a complete simply connected Riemannian manifold having negative sectional curvatures. In this section we give decay conditions, on the curvature K of M, which guarantee that the associated Laplacian Δ has no singular continuous spectrum. By the limiting absorption principle, it suffices to show that the resolvent $R(x) = (\Delta - z)^{-1}$ has good upper and lower boundary values $R^+(z)$, $R^-(z)$ on the real axis. Actually, we will extend $R(z)$ across the real axis, except for a countable set of values which may cluster only at $(n-1)^2/4$.

If M_0 is the simply connected complete space having constant curvature -1, then the special functions results of §2 allow us to continue the resolvent, $R_0(z) = (\Delta_0 - z)^{-1}$. In fact, fix a point $\alpha \in ((n-1)^2/4, \infty)$ and a sufficiently small relatively compact open neighborhood U_α of α. Since $\alpha \in \operatorname{Spec}(\Delta_0)$, $R_0(z)$ cannot be continued from the upper half-plane to U_α as an operator $L^2M_0 \to L^2M_0$. However, let us introduce the weighted spaces

$$L^{2,s}(M_0) = \left\{ f(x) \mid \int_{M_0} |f(x)|^2 e^{2s\gamma(x)}\, dx < \infty \right\}$$

where $\gamma(x) = r(x,p)$, the geodesic distance from a fixed $p \in M_0$. Of course, for $s > 0$, $L^{2,s} \subset L^2 \subset L^{2,-s}$.

One has

LEMMA 5.1. *Let $s > 0$ and $\alpha \in ((n-1)^2/4, \infty)$ be given. Then the resolvent $R_0(z)$ extends from the upper half-plane to a neighborhood U_α of α, as a bounded operator $R_0^+(z)$: $L^{2,s}(M_0) \to L^{2,-s}(M_0)$.*

PROOF. In §2, we obtained a kernel $R_0(z, x, y)$ depending only on z and the geodesic distance $r(x, y)$. The kernel represented $R_0(z)$ for $z \in C - [(n-1)^2/4, \infty)$. Moreover, $R_0(z, x, y)$ extended to the whole z plane with a branch cut along $[(n-1)^2/4, \infty)$.

Choose a smooth function $\chi(x, y) = \chi(r(x, y))$, with $\chi(r) = 1$ for $r < \frac{1}{2}$, and $\chi(r) = 0$ for $r > 1$. We may write $R_0(z, x, y) = \chi(x, y)R_0 + (1 - \chi)R_0$. Although $\chi(x, y)R_0$ has a singularity on the diagonal, it follows from standard properties of pseudodifferential operators that $\chi(x, y)R_0$ defines a bounded operator $L^2M_0 \to L^2M_0$ [**12**, pp. 110–112]. Therefore χR_0 certainly extends to U_α as a bounded operator $L^{2,s}(M_0) \to L^{2,-s}(M_0)$.

The more interesting part of the proof involves $R_1(z, x, y) = (1 - \chi)R_0(z, x, y)$. From (2.4) and the ensuing discussion, we have the estimate

$$R_1(z, x, y) = O\left(\left|e^{[-(n-1)/2 + \sqrt{-1}\, p]r(x,y)}\right|\right)$$

where $z = (n-1)^2/4 + p^2$ and $p > 0$ for $z > (n-1)^2/4$.

Denote

$$R_2(z, x, y) = \exp(-s\gamma(x))R_1(z, x, y)\exp(-s\gamma(y)).$$

Then R_2 is the kernel associated to R_1 via the natural identification $\exp(s\gamma(x))$: $L^{2,s} \to L^2$. It suffices to show that R_2 extends as a bounded operator $L^2 \to L^2$. However, by the triangle inequality,

$$R_2(z, x, y) = O(e^{-[(n-1)/2]r(x,y) - [\gamma(x) + \gamma(y)]s/2})$$

for $z \in U_\alpha$, and U_α sufficiently small.

Now fix $p \in M_0$ and choose geodesic spherical coordinates (r, ω) about p. In these coordinates, the measure $dx = (\sinh r)^{n-1}\, dr\, d\omega$. Denote

$$R_3(z, x, y) = (\gamma^{-1} \sinh \gamma(x))^{(n-1)/2} R_2(z, x, y)(\gamma^{-1} \sinh \gamma(y))^{-(n-1)/2}$$

to be the operator associated to R_2 through the natural map

$$(\gamma^{-1} \sinh \gamma)^{(n-1)/2}: L^2(M_0, dx) \to L^2\left(T_pM_0, r^{n-1}\, dr\, d\omega\right).$$

Then, by the triangle inequality,

$$R_3(z, x, y) = O(e^{-[\gamma(x) + \gamma(y)]s/2}).$$

It suffices to show that R_3 extends to U_α as a map on $L^2(T_pM_0, r^{n-1}\, dr\, d\omega)$. However, the kernel R_3 is Hilbert-Schmidt, so it actually defines a compact operator.

We now transplant the kernel $R_0(z, x, y)$ from M_0 to M and define $S(z, x, y) = R_0(z, r(x, y))$, where $r(x, y)$ is the geodesic distance on M. Denote

$$L^{2,s}(M) = \left\{ f(x) \mid \int_M |f(x)|^2 e^{2s\gamma(x)}\, dx < \infty \right\}$$

where dx is the natural measure of M.

Suppose that the curvature K of M satisfies the decay condition

$$|K(x, \omega) + 1| \leqslant h(x) \tag{5.2}$$

for ω a two-plane in T_xM. We denote $h(x) = C_1 \exp(-C_2\gamma(x))$ for $C_2 > 0$. Then, as observed in [**6**, pp. 8–10], the ratio $|\theta(r, \omega)/\theta_0(r)|$ of volume elements in spherical normal coordinates is bounded above and below by positive constants. Then the proof of Lemma 5.1 shows that $S(z, x, y)$ extends across the real axis to define an operator $S^+(z)$: $L^{2,s}(M) \to L^{2,-s}(M)$, $s > 0$.

It will be important to study the operator with kernel

$$Q(z, x, y) = \left(\frac{-\theta_0'}{\theta_0}(r(x, y)) + \frac{\theta'}{\theta}(x, y)\right)\frac{\partial}{\partial r}S(z, r(x, y)). \tag{5.3}$$

Recall that $z = (n-1)^2/4 + p^2$ with $\sqrt{-1}\,p < 0$ for $z < (n-1)^2/4$.

We have

LEMMA 5.4. *Suppose that the curvature K satisfies the decay conditions* (5.2) *with* $C_2 > 0$. *Let* $0 < s < \min(n-1, C_2)/2$. *Then*

(i) *The kernel $Q(z, x, y)$ defines a compact operator $L^{2,s}(M) \to L^{2,s}(M)$ for $z \in C - [(n-1)^2/4, \infty)$.*

(ii) *Given $\alpha \in ((n-1)^2/4, \infty)$, the operator $Q(z)$ extends from the upper half z plane to a neighborhood U_α of α, as a compact operator $Q^+(z)$: $L^{2,s}(M) \to L^{2,s}(M)$.*

PROOF. Let $P(z, x, y) = \overline{Q(z, y, x)}$. The kernel P is the formal adjoint of Q on $C_0^\infty(M)$. Since compactness is preserved under taking adjoints, it suffices to show that P defines a compact operator $L^{2,-s}(M) \to L^{2,-s}(M)$.

Denote

$$P_1(z, x, y) = \exp(-s\gamma(x))P(z, x, y)\exp(s\gamma(y)).$$

Since compactness is preserved under composition with bounded operators, we need only show that P_1: $L^2M \to L^2M$ defines a compact operator.

Define $\chi(r)$ to be a smooth function satisfying $\chi(r) = 1$ for $r < \frac{1}{2}$ and $\chi(r) = 0$ for $r > 1$. Denote $\chi(x, y) = \chi(r(x, y))$. Then we may write $P_1 = P_2 + P_3$ where $P_2 = \chi P_1$ and $P_3 = (1-\chi)P_1$.

According to Lemma 4.2, the quantity

$$\chi(x, y)|(\theta'/\theta)(x, y) - (\theta_0'/\theta_0)(r(x, y))|e^{-s\gamma(x)}e^{s\gamma(y)}$$

is bounded and approaches zero for $\gamma(x)$ or $\gamma(y)$ large. Here we employ the condition $s < C_2/2$. Using a standard lemma on pseudodifferential operators [**12**, pp. 110–112] and the definition (5.3) of Q, we see that P_2: $L^2M \to L^2M$ is compact. This follows essentially from Rellich's lemma.

Now consider $P_3(z, x, y)$. For $r(x, y) \geqslant \frac{1}{2}$, one has the estimate

$$|(\partial/\partial r)S(z, r(x, y))| = O\left(\left|e^{[-(n-1)/2 + \sqrt{-1}\,p]r(x,y)}\right|\right)$$

which follows from (2.4) and standard asymptotic formulas involving Legendre functions [**17**, pp. 221–222].

If $z \in C - ((n-1)^2/4, \infty)$, then $\mathrm{Im}(p) > 0$. Moreover, in Lemma 5.4(ii), if U_α is sufficiently small, one has $\mathrm{Im}(p) > -\delta$, for any $\delta > 0$. Thus, we may assume that

$$|(\partial/\partial r)S(z, r(x, y))| = O(e^{[-(n-1)/2 + \delta]r(x,y)}) \tag{5.5}$$

where $\delta > 0$ can be forced to be arbitrarily small.

Let $h(x, y)$ be the characteristic function of the set $\{(x, y) | r(x, y) > (1 - \varepsilon)\gamma(x)\}$, where $0 < \varepsilon < 1$ will be specified later. One has $P_3 = P_4 + P_5$, with $P_4 = (1 - h)P_3$ and $P_5 = hP_3$.

For P_4, one may use (5.3), (5.5), and Lemma 4.2 to yield the estimate

$$|P_4(z, x, y)| = O(e^{-[s+\varepsilon C_2]\gamma(x)} e^{[-(n-1)/2+\delta]r(x,y)} e^{s\gamma(y)}).$$

Introducing spherical coordinates (r, ω) about p, recall that $|\theta(r, \omega)/(\sinh r)^{n-1}|$ is bounded above and below by positive constants, as a consequence of our curvature decay conditions [6, pp. 8–10]. We may identify $L^2(M, dx)$ and $L^2(T_pM, r^{n-1}\, dr\, d\omega)$ via $f \to f[\theta(r, \omega)/r^{n-1}]^{1/2}$. Then the kernel P_6: $L^2(T_pM) \to L^2(T_pM)$, associated to P_4, is of order

$$|P_6(z, x, y)| = O(e^{[(n-1)/2 - s - \varepsilon C_2]\gamma(x)} e^{[-(n-1)/2+\delta]r(x,y)} e^{[-(n-1)/2+s]\gamma(y)}).$$

Using the triangle inequality $\gamma(x) \leqslant r(x, y) + \gamma(y)$ yields

$$|P_6(z, x, y)| = O(e^{-\varepsilon C_2 \gamma(x)} e^{(\delta - s) r(x,y)}).$$

If $\delta < s$ and $0 < \mu < s - \delta$, then applying the triangle inequality $\gamma(y) \leqslant \gamma(x) + r(x, y)$, one obtains

$$|P_6(z, x, y)| = O(e^{(-\varepsilon C_2 + \mu)\gamma(x)} e^{-\mu\gamma(y)}).$$

Choosing $\mu < \varepsilon C_2$, we see that P_6: $L^2(T_pM) \to L^2(T_pM)$ is Hilbert-Schmidt and consequently compact. Since compactness is preserved under composition with bounded operators, P_4: $L^2M \to L^2M$ is compact.

Finally, we must deal with $P_5 = hP_3$. Now, the curvature of M is bounded below and, thus [3, p. 284], $|\theta/\theta(x, y) - \theta_0'/\theta_0(r(x, y))|$ is bounded on $M \times M$. Combining this observation with (5.2) and (5.5), one finds

$$|P_5(z, x, y)| = O(h(x, y) e^{-s\gamma(x)} e^{[-(n-1)/2+\delta]r(x,y)} e^{s\gamma(y)}).$$

Using the isomorphism between $L^2(T_pM)$ and L^2M as above, we identify P_5 with an operator P_7: $L^2(T_pM) \to L^2(T_pM)$. The kernel P_7 satisfies

$$|P_7(z, x, y)| = O(h(x, y) e^{[(n-1)/2 - s]\gamma(x)} e^{[-(n-1)/2+\delta]r(x,y)} e^{[-(n-1)/2+s]\gamma(y)}).$$

One may choose ε, δ sufficiently small so that

$$(1 - \varepsilon)((n - 1)/2 - \delta) - ((n - 1)/2 - s) = \beta > 0.$$

Recalling that $h(x, y)$ is the characteristic function of the set $\{(x, y) | r(x, y) > (1 - \varepsilon)\gamma(x)\}$, we find

$$|P_7(z, x, y)| = O(e^{-\beta\gamma(x)} e^{[-(n-1)/2+s]\gamma(y)}).$$

Then P_7: $L^2(T_pM) \to L^2(T_pM)$ is Hilbert-Schmidt. Consequently, P_5 is compact.

This completes the proof of Lemma 5.4.

We may now extend the resolvent kernel of M:

PROPOSITION 5.6. *Let $0 < s < \min(n - 1, C_2)/2$ be given. Suppose that $\alpha \in R - \Lambda$, where Λ will be some countable set of points which may cluster only at $(n - 1)^2/4$. Then the resolvent $R(z) = (\Delta - z)^{-1}$ extends from the upper half-plane to a neighborhood U_α of α as a bounded operator $R^+(z)$: $L^{2,s} \to L^{2,-s}$.*

PROOF. Fix $\alpha \in R - (n-1)^2/4$ and U_α so that the conclusion of Lemma 5.4 is satisfied. One has the second resolvent equation

$$S^+(z) = R(z)[I + Q^+(z)].$$

From Lemma 5.4, we see that $I + Q^+(z)$: $L^{2,s} \to L^{2,s}$ is Fredholm. Using a standard lemma on families of compact operators [**14**, p. 370] we see that $\ker[I + Q^+(z)] = 0$ except for finitely many $z \in \Lambda_\alpha \subset U_\alpha$. For $z \in U_\alpha - \Lambda_\alpha$, the Fredholm alternative allows us to continue $R(z)$ by

$$R^+(z) = S^+(z)[I + Q^+(z)]^{-1}.$$

Our main result for the present section is

THEOREM 5.7. *Let M be a complete simply connected negatively curved manifold. Suppose that $K(x, \pi)$ denotes the sectional curvature of the two-plane π in T_xM. Impose the curvature decay condition $|K(x, \pi) + 1| \leqslant C_1 \exp(-C_2\gamma(x))$, where $\gamma(x) = r(x, p)$ is the geodesic distance from p in M. Then the Laplacian Δ of M has no singular continuous spectrum.*

PROOF. Proposition 5.6 shows that $R(z)$ has good lower boundary values $R^+(z)$: $L^{2,s} \to L^{2,-s}$. Similarly, one shows that there are good upper boundary values $R^-(z)$: $L^{2,s} \to L^{2,-s}$. Theorem 5.7 now follows from the limiting absorption principle [**7**, p. 64], [**8**, p. 1202].

REMARK 5.8. In the author's earlier paper [**7**], it was shown that if Δ is the Laplacian of a metric obtained by a compactly supported perturbation of the metric on the constant curvature space M_0, then Δ has no singular continuous spectrum. Theorem 5.7 is much stronger, since one need only satisfy curvature decay conditions. No restraints are imposed on the other derivatives of the metric g. In fact, M need not be obtained from M_0 by a perturbation of g_0.

6. Identifying the Laplacian up to unitary equivalence. One may combine Theorems 4.12 and 5.7 to obtain a condition guaranteeing stability of the continuous spectrum:

THEOREM 6.1. *Let M be a complete simply connected Riemannian manifold having negative sectional curvatures. Suppose that the metric of M is obtained by perturbation from the standard metric g_0 on the simply connected space of constant curvature -1.*

If g, K denote the metric and curvature of M, then we impose the decay conditions:

(i) $(1 + \beta)^{-2}g_0(V, V) \leqslant g(V, V) \leqslant (1 + \beta)^2 g_0(V, V)$ *for* $V \in T_xM$, *and*

(ii) $|K(x, \pi) + 1| \leqslant h$ *for π a two-plane at $x \in M$.*

Here $h(x) = C_1 \exp(-C_2\gamma(x))$ and $\beta(x) = C_3 \exp(-C_2\gamma(x))$. Moreover, $\gamma(x) = r(x, p)$ is the geodesic distance of x from a fixed $p \in M$. We assume that $C_2 > n - 1$, where n is the dimension of M.

Under these conditions the continuous part of the Laplacian Δ: $L^2M \to L^2M$ is unitarily equivalent to Δ_0: $L^2M_0 \to L^2M_0$.

The eigenvalues λ embedded in the continuum, that is $\lambda > (n-1)^2/4$, were discussed in the author's earlier paper [**6**].

THEOREM 6.2. *Suppose that conditions* (i) *and* (ii) *of Theorem* 6.1 *are satisfied, and also*

(iii) $\int_0^\infty \|\nabla_\omega K\| e^{2r}\, dr < D_1$,

(iv) $\int_0^\infty \|\nabla_\omega^2 K\| e^{2r}\, dr < D_2$

for some constants $D_1, D_2 > 0$. *Here* $\nabla_\omega K$ *is the covariant derivative in geodesic spherical coordinates* (r, ω) *of any sectional curvature* K *along the geodesics emanating from* p.

Then Δ *has the same continuous part as* Δ_0. *Moreover,* Δ *has no eigenvalue* $\lambda > (n-1)^2/4$.

Finally, by adding one more condition, we obtain a stability theorem for the entire spectrum:

THEOREM 6.3. *Let* M *be as in Theorem* 6.2 *and assume that*

(v) $K \leqslant -1$.

Then Δ *is unitarily equivalent to* Δ_0.

PROOF. If $K \leqslant -1$ then the spectrum of Δ is bounded below by $(n-1)^2/4$ [**16**, I, p. 88], [**18**, p. 498]. Moreover, $(n-1)^2/4$ cannot occur as an eigenvalue [**6**, p. 11], [**16**, II, p. 4] when $K \leqslant -1$.

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